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Coherent states for anharmonic oscillator Hamiltonians with equidistant and quasi-equidistant spectra

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Abstract. Two kinds of transformation for the time-dependent Schrödinger equation, i.e. the differential and integral transformations, are introduced. If one considers only stationary solutions of this equation, both transformations reduce to the well known Darboux transformation for the stationary Schrödinger equation. When applied to non-stationary solutions, they give different results. Both transformations are invertible in appropriate spaces. With the help of these transformations alternative systems of coherent states to those in the literature are obtained for isospectral Hamiltonians with equidistant spectra. These transformations are also applied to the construction of coherent states for Hamiltonians whose spectrum consists of an equidistant part and one separately disposed level with an energy gap equal to the k skipped levels.

1. Introduction

Recently [1, 2] the coherent states for isospectral oscillator Hamiltonians, previously obtained in [3] and investigated in [4], have been studied. Another type of Hamiltonian, namely Hamiltonians with quasi-equidistant spectra (e.g., a spectrum with lacunae) also exists. It can readily be seen that both types of Hamiltonian can be obtained from the harmonic oscillator Hamiltonian through the known Darboux transformation [5].

The coherent states are non-stationary solutions of the time-dependent Schrödinger equation. To obtain the solutions of this equation, one can combine the ordinary Darboux transformation operator technique with the propagator operator technique [6]. In this paper, to obtain the solutions of one time-dependent Schrödinger equation, if the solutions of another one are known, we introduce two types of time-dependent transformation. In the case of stationary solutions, both types of transformation reduce to the Darboux transformation; however, being applied to non-stationary solutions, they give different results. One can consider both types of transformation as a generalization of the Darboux transformation to the time-dependent Schrödinger equation. This approach is simpler to apply to concrete calculations than the propagator operator technique, especially when the Hamiltonian depends on time.

The transformations introduced, being invertible, permit one to construct a symmetry algebra for the new Schrödinger equation, if the symmetry algebra for the initial one is known. In particular, if for the initial Schrödinger equation one knows such integrals of motion as the creation and annihilation operators, similar operators can be constructed for the new Schrödinger equation, and their eigenstates can be obtained by merely acting on the eigenstates of the initial operator by the transformation operator. In this manner, one can obtain different systems of coherent states for the new Hamiltonians, if the coherent states for

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the initial one are known. In this paper alternative systems of coherent states to those given in [1] are obtained for isospectral oscillator Hamiltonians and for the anharmonic oscillator Hamiltonians with quasi-equidistant spectra. A comparison of the obtained coherent states with those of [2] is made.

The plan of the paper is as follows. In section 2, we introduce the time-dependent transformation operators and investigate some of their properties. In section 3, we give briefly the known properties of the coherent states for the harmonic oscillator potential. Then we construct one class of potentials with quasi-equidistant spectra, and for this class as well as for the isospectral oscillator Hamiltonians we construct the systems of coherent states.

2. Transformation operators

Let us consider the time-dependent Schrödinger equation for a particle with a potential energy $-V_0(x,t)$:

$$[\mathrm{i}\partial_t + \partial_x^2 + V_0(x,t)]\psi(x,t) = 0 \qquad \partial_t = \partial/\partial_t \qquad \partial_x^2 = \partial_x \cdot \partial_x \qquad x \in [a,b]. \tag{1}$$

We say that an operator L acting on the solutions of equation (1) is a transformation operator if the following equality holds:

$$L(\mathrm{i}\partial_t + \partial_x^2 + V_0(x,t)) = (\mathrm{i}\partial_t + \partial_x^2 + V_1(x,t))L. \tag{2}$$

The operator L defined by this relation will transform the solutions of equation (1) into the solutions of another Schrödinger equation, $\varphi(x,t) = L\psi(x,t)$:

$$[\mathrm{i}\partial_t + \partial_x^2 + V_1(x,t)]\varphi(x,t) = 0 \qquad x \in [a,b]. \tag{3}$$

2.1. Differential transformation operators

Let the operator L be a first-order differential operator of the form

$$L = L_0(x,t) + L_1(x,t)\partial_x. \tag{4}$$

We do not include in (4) the derivative with respect to t since it, being found from equation (1), transforms L into the second-order differential operator, but we will restrict ourselves by the first-order differential operators. Equation (2) results in a system of equations for the functions $L_0(x,t)$, $L_1(x,t)$, and the potential difference

$$A(x,t) = H_1 - H_0 = V_1(x,t) - V_0(x,t) \tag{5}$$

$$L_{1x} = 0 \qquad \mathrm{i}L_{1t} + 2L_{0x} = -AL_1 \qquad L_1V_{0x} - \mathrm{i}L_{0t} - L_{0xx} = AL_0. \tag{6}$$

We use here the conventional notation for derivatives: $f_x = \partial f/\partial x$, $f_{xx} = \partial^2 f/\partial x^2$, $f_x^{(k)} = \partial^k f/\partial x^k$, and omit the arguments of the functions if this does not cause a misunderstanding. The latter system can readily be integrated with $C = C(t)$ as a constant of integration, and we obtain for a function $u = u(x,t)$ defined by the relation $u_x/u = -L_0/L_1$ the following equation:

$$\mathrm{i}u_t + u_{xx} + (V_0 - C)u = 0.$$

The function $L_1 = L_1(t)$ remains arbitrary, and the quotient $r = L_0/L_1$ does not depend on the constant C and, consequently, it can be calculated for $C = 0$. The function $u(x,t)$, called a transformation function, is in this case a solution of the initial Schrödinger equation (1).

For the potential difference $A(x,t)$ we obtain $A = -\mathrm{i}(\log L_1)_t - 2r_x$. The function $L_1(t)$ can be chosen such that it ensures the reality condition for the function A. This

condition, when presented in the form $\mathrm{i}(\log|L_1|^2)_t = 2(r_x^* - r_x)$, results in the condition for the transformation function u

$$\left(\log\frac{u}{u^*}\right)_{xxx} = 0 \tag{7}$$

where the asterisk implies complex conjugation. If the function L_1 is chosen as real,

$$L_1 = \exp\left(2\int \mathrm{d}t\ \mathrm{Im}(\log u)_{xx}\right) \tag{8}$$

the transformation function $u(x,t)$ completely defines the potential difference

$$A = (\log|u|^2)_{xx} \tag{9}$$

and the transformation operator

$$L = L_1 u^{-1}\begin{vmatrix} u & 1 \\ u_x & \partial_x \end{vmatrix} = L_1(-u_x/u + \partial_x). \tag{10}$$

The operator determinants, such as those in formula (10), must be considered as differential operators obtained by the development of the determinants in the latter column with the functional coefficients placed before the derivative operators.

The action of the operator L on the transformation function u gives zero. However, if the transformation function u is chosen in accordance with formula (7), we can readily verify that the function

$$\varphi_0 = \frac{1}{L_1 u^*} \tag{11}$$

is a solution of the new Schrödinger equation (3).

If we use this function as the transformation function for the second transformation, then the new potential difference differs from the one previously obtained by sign, and we return to the initial Schrödinger equation. The transformation operator for this case is $L^+ = -L_1(u_x^*/u^* + \partial_x)$. It is clear that the successive action of the operators L and L^+ transforms, in the general case, one solution of equation (1) into another solution, and consequently L^+L is a symmetry operator for this equation. In full analogy, the operator LL^+ is a symmetry operator for equation (3).

Let T_0 and T_1 be the linear spaces of the solutions of equations (1) and (3), respectively, defined over the complex number field $\mathcal{C}$. It follows from formula (10) that $\mathrm{Ker}\,L$ is a one-dimensional space spanned on the transformation function u and the set of functions $\{\varphi: \varphi = L\psi,\ \psi \in T_0\}$ does not span all space T_1. We shall consider later the initial potential as a stationary one, $V_0 = V_0(x)$, and choose the transformation function as a stationary solution of the initial equation (1): $u(x,t) = u_\alpha(x)\exp(-\mathrm{i}\alpha t)$, $H_0 u_\alpha(x) = \alpha u_\alpha(x)$ where $H_0 = -\partial_x^2 - V_0(x)$ is the initial Hamiltonian. In this case $L_1 = 1$ and the prototype of function φ_0 (11) designated by $\tilde{u}(x,t) = \tilde{u}_\alpha(x)\exp(-\mathrm{i}\alpha t)$ corresponds to the second proper function of the Hamiltonian H_0 with the same proper value α. This function can be found from the condition for the Wronskian $W(u_\alpha, \tilde{u}_\alpha) = 1$:

$$\tilde{u}_\alpha(x) = u_\alpha(x)\int u_\alpha^{-2}\,\mathrm{d}x \qquad L\tilde{u}(x,t) = \varphi_0(x,t).$$

In the majority of cases, the potentials $V_0(x,t)$ being of physical interest are such that it is possible to introduce a Hilbert space structure $L_0^2(R)$ (the interval of variable x for equation (1), $R = [a,b]$, can span the whole real axis) in the space T_0 with the scalar product appropriately defined. In this case, an integral of motion usually exists for equation (1), which is a self-adjoint second-order differential operator with a discrete spectrum. The set

of its discrete eigenfunctions $\psi_i(x,t)$ forms a discrete basis set of the space $L_0^2(R)$. The functions $\psi_i(x,t)$ are usually ordered in such a way that the number i of each function coincides with its number of zeros. If we demand the transformation function u to be nodeless for all t and $x \in [a,b]$, the potential difference $A(x,t)$ will be a regular function on $[a,b]$ and the transformation operator L will be well defined. There exists an unique (up to a constant factor) nodeless function in $L_0^2(R)$. In contrast, beyond the space $L_0^2(R)$, there are many nodeless functions suitable for use as transformation functions (for the stationary case see [7, 8]). In this paper we shall consider the restriction of operator L on the space $L_0^2(R)$ and designate it by the same symbol L. We shall also suppose that $\varphi_0 = 1/(L_1 u^*) \in L_1^2(R)$ where $L_1^2(R)$ is the space of the square integrable on the interval R solutions of equation (3). In this case $\tilde{u} \notin L_0^2(R)$.

In our case the space $L_0^2(R)$ is invariant under the action of the symmetry operator L^+L, and the space $\tilde{L}_1^2(R)$, defined as the image of the space $L_0^2(R)$ induced by operator L, is included in the space $L_1^2(R)$. The space $L_1^2(R)$ is a join of the space $\tilde{L}_1^2(R)$ and the linear hull spanning the function φ_0. The system $\{(L_1 u^*)^{-1}, L\psi_i\}$ forms a basis in the space $L_1^2(R)$, in the case $\{\psi_i\}$ is a basis in $L_0^2(R)$.

Under the aforesaid hypotheses for the function u, we have for $\psi \in L_0^2(R)$ the equality $\int_{-\infty}^{x} u^{-1} L\psi \, \mathrm{d}x = u^{-1}\psi$. It follows that for every $\varphi \in \tilde{L}_1^2(R)$ we can define an integral transformation operator M

$$M\varphi = u \int_{-\infty}^{x} u^{-1}\varphi \, \mathrm{d}x. \tag{12}$$

We can readily verify that $ML\psi = \psi$ and $LM\varphi = \varphi$, e.g., operator M is inverse to L. It is necessary to note that $M\varphi_0(x,t) = \tilde{u}(x,t) \notin L_0^2(R)$ if we define the action of the operator M on the function φ_0 by the same formula.

Transformation (4), if repeated N times, leads to an N-order transformation operator of the form

$$L^{(N)} = L_N(t) W^{-1}(u_1, u_2, \ldots, u_N) \begin{vmatrix} u_1 & u_2 & \cdots & 1 \\ u_{1x} & u_{2x} & \cdots & \partial_x \\ \vdots & \vdots & \ddots & \vdots \\ u_{1x}^{(N)} & u_{2x}^{(N)} & \cdots & \partial_x^{(N)} \end{vmatrix} \tag{13}$$

and a potential difference of the form

$$A_N = (\log |W(u_1, u_2, \ldots, u_N)|^2)_{xx} \tag{14}$$

if the transformation functions $u_1, u_2, \ldots, u_n$ satisfy the reality condition

$$\left[\log \frac{W(u_1, u_2, \ldots, u_N)}{W^*(u_1, u_2, \ldots, u_N)} \right]_{xxx} = 0. \tag{15}$$

For the real function $L_N(t)$ we have

$$L_N(t) = \exp 2 \int \mathrm{d}t \, \mathrm{Im}[\log W(u_1, u_2, \ldots, u_N)]_{xx}. \tag{16}$$

Here we use the conventional symbol $W(u_1, u_2, \ldots, u_N)$ to denote the Wronskian of functions $u_1, u_2, \ldots, u_N$.

The operator $L^{(N)}$, being applied to the transformation function u_k, gives null. Nevertheless, for the transformation function subjected to the reality conditions (7) and (15), we have the following solutions of equation (3) with potential difference (14):

$$\varphi^{(k)}(x,t) = \frac{W^{(k)^*}(u_1, \ldots, u_N)}{L_N(t) W^*(u_1, \ldots, u_N)}$$

where $W^{(k)}(u_1, \ldots, u_N)$ is the Wronskian of the functions $u_1, \ldots, u_N$ except u_k.

If the transformation function $u = u_1 = \psi_n \in L_0^2(R)$ has n zeros, then potential difference (9) has n poles and the solutions obtained with the help of transformation operator (10), $L\psi_i$, does not belong to $L_1^2(R)$. Nevertheless, the second transformation with the transformation function $u_2 = \psi_{n+1} \in L_0^2(R)$ removes all singularities and the transformation operator $L^{(2)}$ (13) is well defined. This fact reflects the known properties of the Wronskians constructed from the functions belonging to $L_0^2(R)$ [9]: the Wronskian $W(u_{k_1}, \ldots, u_{k_N})$ conserves its sign for $u_{k_i} \in L_0^2(R)$ if for all $k = 0, 1, 2, \ldots$, the numbers k_i being equal to the number of zeros of the functions u_{k_i}, the inequality $(k - k_1)(k - k_2) \ldots (k - k_N) \geqslant 0$ holds. In particular, the functions u_{k_i} may be two by two juxtaposed functions.

If the transformation functions are stationary solutions of equation (1) of the form $u(x, t) = u_\alpha(x) \cdot \exp(-i\alpha t)$, then the potential difference (14) does not depend on time, the reality condition (15) holds for all transformation functions, and formulae (13) and (14) reduce to the known formulae of the Darboux transformation repeated N times [9, 10]. It is not difficult to establish that the symmetry operator L^+L is a polynomial function of the initial Hamiltonian [11]: $L^+L = (H_0 - \alpha_1)(H_0 - \alpha_2) \ldots (H_0 - \alpha_N)$, and LL^+ is the same function of the final Hamiltonian: $LL^+ = (H - \alpha_1)(H - \alpha_2) \ldots (H - \alpha_N)$. This property forms the basis of the known factorization method [12, 13]. To obtain the time-dependent solutions of the nonstationary Schrödinger equation, we can also use in this case the propagator operator technique combined with the usual Darboux transformation. This approach is realized in [6].

2.2. Integral transformation operators

Integral transformations for the stationary Schrödinger equation are well studied in the theory of the inverse scattering problem (see, e.g., [14, 15]) and in its application to solving the nonlinear equations [16]. The connection of these transformations with the Darboux transformation is, in particular, discussed in [11, 17]. The possibility of an integral representation of the Darboux transformations is noted by Faddeev [18]. This possibility is due to the fact that the Wronskian of functions ψ_1 and ψ_2 can be calculated with the help of the integral

$$W(\psi_1, \psi_2) = (E_2 - E_1) \int^x \psi_1 \psi_2 \, \mathrm{d}x + C.$$

The integral transformation can be generalized to the time-dependent Schrödinger equation. One can find one of these generalizations in [19, 20]. In this section we introduce integral transformations, more general in some sense, for the time-dependent Schrödinger equation.

It is intuitively clear that the transformation inverse to the differential one (if it is invertible) must be an integral one. On the other hand, the initial and final Schrödinger equations are equivalent, and it is unimportant whether one starts from equation (1) or from equation (3). Therefore, the operator inverse to L will, in turn, be a transformation operator for equation (3).

To clarify these ideas, let us consider some detail in the definition of the operator inverse to the differential operator L of the preceding section. Operator L being applied to a function $f(x)$, gives another function

$$F(x) = -\frac{u_x}{u} f(x) + f_x(x). \tag{17}$$

If we know the function F, we can readily find the function f from differential equation (17):

$$f(x) = u\left(C + \int_{x_0}^{x} u^{-1}F(x)\,\mathrm{d}x\right). \tag{18}$$

It is clear that we have indeed obtained a family of functions f. In this formula, function u is a solution of equation (1) and function F is the solution of another equation, namely (3). In order that the right-hand side of relation (18) be expressed through the solutions of the same equation, we have (according to relation (11)) to replace $u \to 1/(L_1u^*)$. To define correctly the integral transformation operator, it remains now to take into consideration the dependence of all functions on time t.

Let functions $u(x,t)$ and $\psi(x,t)$ be solutions of equation (1). Define a function $w(u^*(x,t), \psi(x,t)) \equiv w(u^*,\psi) = w(x,t)$ by the following relations:

$$w_x(u^*,\psi) = u^*\psi \tag{19}$$

$$w_t(u^*,\psi) = \mathrm{i}(u^*\psi_x - u_x^*\psi). \tag{20}$$

Using equation (1) we can readily establish that $w_{xt} = w_{tx}$ and the function w is well defined by relations (19) and (20). Hence it follows that the function w can be calculated in two different manners:

$$w = \mathrm{i}\int_{t_0}^{t} (u^*\psi_x - u_x^*\psi)\,\mathrm{d}t + C_1(x) \tag{21}$$

$$w = \int_{x_0}^{x} u^*\psi\,\mathrm{d}x + C_2(t). \tag{22}$$

For the integration constants $C_1(x)$ and $C_2(t)$, using (19), (20) and equation (1), we find

$$C_1'(x) = (u^*\psi)|_{t=t_0} \qquad C_2'(t) = \mathrm{i}(u^*\psi_x - u_x^*\psi)|_{x=x_0}.$$

Let function u, named transformation function, be a fixed solution of equation (1). Also let space $T_{01} \subset T_0$ be a linear space of functions ψ such that $C_1'(x) = 0$. Designate as $T_{02}(\subset T_0)$ a linear space of functions ψ such that $C_2'(t) = 0$. For every space T_{01} and T_{02} we can define an integral transformation operator M as follows:

$$\varphi(x,t) = M\psi(x,t) = \frac{w(u^*,\psi)}{L_1(t)u^*} \tag{23}$$

where function $w(u^*,\psi)$ is defined by formula (21) for the space T_{01} and by formula (22) for T_{02}. The straightforward calculation persuades us that function (23) is a solution of the new Schrödinger equation (3) if the transformation function u obeys the reality condition (7) and the potential difference A and function $L_1(t)$ are calculated by formulae (9) and (8), respectively. By virtue of the fact that function φ_0 (11) satisfies the same equation (3), the constants C_1 and C_2 from formulae (21) and (22) can be put equal to zero.

Note that if the transformation function u is such that u^{-1} is square integrable function (as it was supposed in subsection 2.1), then $L_0^2(R) \subset T_{02}$. This is the reason that the transformation operator (12) is defined by formulae (22) and (23) at $L_1(t) \equiv 1$ and with the replacement $u^* \to u^{-1}$.

Consider for definiteness the space T_{01}. Operator M induces in T_1 a subspace $T_{11} = \{\varphi: \varphi = M\psi,\ \psi \in T_{01}\}$. Let us define in the space T_{11} a differential operator L of the form

$$L = L_1(t)(u_x^*/u^* + \partial_x) \tag{24}$$

which, obviously, in terms of equation (3) is a differential transformation operator for this equation and, consequently, it obeys the equation

$$(\mathrm{i}\partial_t + \partial_x^2 + V_0(x,t))L = L(\mathrm{i}\partial_t + \partial_x^2 + V_1(x,t)).$$

It is not difficult to establish that $L\varphi = LM\psi = \psi$ and $M\psi = ML\varphi = \varphi$, e.g., operator L (24) is inverse to M, which obeys equation (2).

Formula (23), by analogy with the differential transformation, can be generalized to N transformation functions of the form

$$\varphi(x,t) = \frac{w(\psi, u_1, \ldots, u_N)}{L_N(t) w(u_1, \ldots, u_N)}. \tag{25}$$

The definition of the function $w(u_1, \ldots, u_N)$ depends on the parity of N. If N is even, then we have

$$w(u_1, \ldots, u_N) = \begin{vmatrix} w(u_2^*, u_1) & w(u_4^*, u_1) & \cdots & w(u_N^*, u_1) \\ w(u_2^*, u_3) & w(u_4^*, u_3) & \cdots & w(u_N^*, u_3) \\ \vdots & \vdots & \ddots & \vdots \\ w(u_2^*, u_{N-1}) & w(u_4^*, u_{N-1}) & \cdots & w(u_N^*, u_{N-1}) \end{vmatrix} \tag{26}$$

and

$$w(\psi, u_1, \ldots, u_N) = \begin{vmatrix} \psi & w(u_2^*, \psi) & w(u_4^*, \psi) & \cdots & w(u_N^*, \psi) \\ u_1 & w(u_2^*, u_1) & w(u_4^*, u_1) & \cdots & w(u_N^*, u_1) \\ u_3 & w(u_2^*, u_3) & w(u_4^*, u_3) & \cdots & w(u_N^*, u_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{N-1} & w(u_2^*, u_{N-1}) & w(u_4^*, u_{N-1}) & \cdots & w(u_N^*, u_{N-1}) \end{vmatrix}. \tag{27}$$

For odd N we have

$$w(u_1, \ldots, u_N) = \begin{vmatrix} u_1^* & w(u_1^*, u_2) & w(u_1^*, u_4) & \cdots & w(u_1^*, u_{N-1}) \\ u_3^* & w(u_3^*, u_2) & w(u_3^*, u_4) & \cdots & w(u_3^*, u_{N-1}) \\ u_5^* & w(u_5^*, u_2) & w(u_5^*, u_4) & \cdots & w(u_5^*, u_{N-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_N^* & w(u_N^*, u_2) & w(u_N^*, u_4) & \cdots & w(u_N^*, u_{N-1}) \end{vmatrix} \tag{28}$$

and

$$w(\psi, u_1, \ldots, u_N) = \begin{vmatrix} w(u_1^*, \psi) & w(u_3^*, \psi) & \cdots & w(u_N^*, \psi) \\ w(u_1^*, u_2) & w(u_3^*, u_2) & \cdots & w(u_N^*, u_2) \\ \vdots & \vdots & \ddots & \\ w(u_1^*, u_{N-1}) & w(u_3^*, u_{N-1}) & \cdots & w(u_N^*, u_{N-1}) \end{vmatrix}. \tag{29}$$

The formulae for the function $L_N(t)$, the potential difference $A_N(x,t)$, and the reality condition coincide with (14), (16), and (15) with the replacement of W by w. The functions $w(u_i^*, u_j)$ from the determinants (26)–(29) are defined by formulae (21) or (22) to constant factors $C_{i,j}$. One can choose these constants to ensure the regularity condition of the potential difference. Similar constants entering the functions $w(u_i^*, \psi)$ can, without loss of generality, be put equal to zero since the new Schrödinger equation also has the following solutions:

$$\varphi^{(k)}(x,t) = \frac{L_N(t) w^{(k)}(u_1, \ldots, u_N)}{w(u_1, \ldots, u_N)}$$

where $w^{(k)}(u_1, \ldots, u_N)$ is to be calculated for even N by formula (28) and for odd N by formula (26) with the set of functions $u_1, \ldots, u_N$ except the function u_k.

For even N, there always exists the possibility of choosing the transformation functions to ensure the reality condition of the function $w(u_1, \ldots, u_N)$. For this purpose, we must assume $u_{2i} = u_{2i-1}$, $i = 1, \ldots, N/2$ and choose the constants $C_{i,j}$ such that the matrix of determinant (26) be Hermitian. Its determinant will then be real. If the transformation

functions satisfy the conditions $u_i(a,t)=0$ and $u_{ix}(a,t)<\infty$, then $C_2'(t)=0$ at $x_0=a$ for all transformation functions u_i and the functions $\psi\in L_0^2(R)$ satisfying the zero boundary conditions. Formulae (22) and (25)–(27) define, in this case, the transformation discussed in [19, 20].

Let us now consider a special case corresponding to the stationary solutions of equation (1) of the form $\psi(x,t)\equiv\psi_E(x,t)=\mathrm{e}^{-\mathrm{i}Et}\psi_E(x)$. The reality condition (7) holds for all stationary transformation functions. For the present instance (e.g., the harmonic oscillator potential) the discrete spectrum of the initial Hamiltonian is bounded below by the value $E=E_{\min}$. It is evident that if $\psi_E(x,t)\in L_0^2(R)$, the transformation function $u_\alpha(x,t)\notin L_0^2(R)$ (in the case $\alpha<E_{\min}$), and $\varphi_0=(L_1u^*)^{-1}\in L_1^2(R)$ then $C_1'(x)=0$ for any ψ_E at $t_0=-\mathrm{i}\cdot\infty$. It follows that $L_0^2(R)\subset T_{01}$ and in the space $L_0^2(R)$ formulae (23) and (21) define an integral transformation operator acting on $\tilde{L}_1^2(R)\subset L_1^2(R)$. This operator cannot directly give the function φ_0 by action on any function $\psi\in L_0^2(R)$. The function w, being defined by relation (21), is proportional to the Wronskian of the functions $u_\alpha(x)$ and $\psi_E(x)$. The action of the operator M on stationary states $\psi_E(x,t)$ gives, to a constant factor, the same result as the differential operator (10). One can obtain from formulae (25)–(27) the integral transformations for the stationary solutions of the Schrödinger equation well known in the theory of inverse scattering. For the non-stationary states $\psi(x,t)$ the integral transformation and the differential one give different results.

It is necessary to note that we can extend the domain of action of the operator M beyond the space T_0 to all functions $\psi(x,t)$ such that the condition $C_1'(x)=0$ holds.

3. Coherent states

Operators L (10) and M (23) are invertible in the corresponding spaces. If g is a symmetry operator for the initial Schrödinger equation, then for the final one we can construct two corresponding independent symmetry operators $\overline{h}=LgL^{-1}$ and $\tilde{h}=MgM^{-1}$. It is clear that if the sets of operators $\{g\}$, $\{\overline{h}\}$, $\{\tilde{h}\}$ form algebras G, $\overline{H}$, $\tilde{H}$, then they are isomorphic. If, however, G is an algebra of differential operators, then $\overline{H}$ and $\tilde{H}$ are algebras of integro-differential operators.

Thus, for creation $\hat{a}^+$ and annihilation $\hat{a}$ operators for an harmonic oscillator we can construct the following operators: $\hat{b}^+=L\hat{a}^+L^+$, $\hat{b}=L\hat{a}L^+$, $\overline{b}^+=L\hat{a}^+L^{-1}$, $\overline{b}=L\hat{a}L^{-1}$, $\tilde{b}^+=M\hat{a}^+M^{-1}$, $\tilde{b}=M\hat{a}M^{-1}$. Operators $\hat{b}^+$ and $\hat{b}$ will be the differential ladder operators for the new Hamiltonian eigenfunctions, but they do not satisfy the appropriate commutation relation and cannot play the role of creation and annihilation operators. Operators $\overline{b}$, $\overline{b}^+$ and $\tilde{b}$, $\tilde{b}^+$ being integro-differential, in contrast, conserve all the properties of annihilation and creation operators. It is clear that functions $\overline{\varphi}_\alpha=L\psi_\alpha$ and $\tilde{\varphi}_\alpha=M\psi_\alpha$ will be proper functions for these operators and, consequently, they can be treated as coherent states for new Hamiltonians if we define the coherent states as proper states of an annihilation operator.

Now we shall give a brief survey of the well known properties of the coherent states of the harmonic oscillator potential needed for our purposes.

3.1. Harmonic oscillator coherent states

Let us consider $V_0(x,t)=V_0(x)=-x^2/4$. The creation and annihilation operators $\hat{a}^+(t)$ and $\hat{a}(t)$ are differential operators belonging to the Lie algebra of the first-order differential symmetry operators of equation (1). This algebra is isomorphic to the symmetry Lie algebra

of the free-particle Schrödinger equation. Its explicit form is as follows:

$$\hat{a}(t) = \mathrm{e}^{\mathrm{i}t}(\partial_x + x/2) \qquad \hat{a}^+(t) = \mathrm{e}^{-\mathrm{i}t}(-\partial_x + x/2) \qquad \hat{a}(t)\hat{a}^+(t) - \hat{a}^+(t)\hat{a}(t) = 1.$$

We define the coherent states as the proper states of the annihilation operator. This corresponds to the solutions of the initial Schrödinger equation with separated variables:

$$\begin{aligned} &\psi_z(x,t) = (2\pi)^{-1/4}\exp(-\tfrac{1}{4}x^2 - \tfrac{1}{2}\mathrm{i}t + zx - \tfrac{1}{2}z^2 - \tfrac{1}{2}zz^*) \\ &\hat{a}(t)\psi_z(x,t) = \zeta\psi_z(x,t) \qquad z = \zeta\mathrm{e}^{-\mathrm{i}t}. \end{aligned} \tag{30}$$

If we wish to have an harmonic oscillator with a frequency ω, we have to perform the changes $x^2 \to 2\omega x^2$ and $t \to 2\omega t$. One can develop the function ψ_z on the set of orthonormal stationary states of the harmonic oscillator $|n;t\rangle = \exp[-\mathrm{i}(n+\frac{1}{2})t]|n\rangle$, complete in $L_0^2(R)$:

$$|z;t\rangle = \exp(-zz^*/2 - \mathrm{i}t/2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle. \tag{31}$$

The coordinate representation of the stationary states is as follows:

$$\begin{aligned} &\psi_n(x,t) = (2\pi)^{-1/4}(n!)^{-1/2}\exp(-x^2/4 - \mathrm{i}(n+\tfrac{1}{2})t)He_n(x) \\ &He_n(x) = 2^{-n/2}H_n\left(x/\sqrt{2}\right) \end{aligned} \tag{32}$$

$H_n(z)$ being the Hermite polynomials [21].

It is not difficult to obtain the coordinate and momentum expectation values and their variances in the states $|z;t\rangle$. The expectation energy value, for example, is equal to $\langle E\rangle = \langle z;t|H_0|z;t\rangle = zz^* + \frac{1}{2}$, where $H_0 = -\partial^2/\partial x^2 - V_0(x)$ is the harmonic oscillator Hamiltonian. The Fock–Bargman representation is very useful for many cases (see, e.g., [22]).

3.2. Anharmonic oscillator potentials

3.2.1. Consider first the isospectral oscillator potentials previously obtained in [3] and then studied in [4, 1]. To obtain these potentials, we use the differential transformation operator (10) and the following solution of the initial Schrödinger equation as a transformation function:

$$u(x,t) = \exp(\mathrm{i}t/2 + x^2/4)\left[C + \mathrm{erf}\left(x/\sqrt{2}\right)\right] \qquad H_0u(x,t) = -\tfrac{1}{2}u(x,t).$$

The potential difference calculated by formula (9) has the form

$$\begin{aligned} &A(x,t) = A(x) = 1 - 2xQ_1^{-1}(x)\exp(-x^2/2) - 2Q_1^{-2}(x)\exp(-x^2) \\ &Q_1(x) = \sqrt{\frac{\pi}{2}}\left(C + \mathrm{erf}\left(x/\sqrt{2}\right)\right). \end{aligned}$$

The potentials are well defined for $|C| > 1$. The stationary-state wavefunctions can be obtained either by integral transformation (23) or by a differential one (10), whose action on the oscillator stationary-state wavefunctions (32) differs by an unessential constant factor. The functions normalized to unity have the following form:

$$\begin{aligned} &\theta_0(x,t) = (\pi/8)^{1/4}\sqrt{C^2-1}\exp(\mathrm{i}t/2 - x^2/4)Q_1^{-1}(x) \\ &\theta_{n+1}(x,t) = (2\pi)^{-1/4}[(n+1)!]^{-1/2}\exp(-\mathrm{i}(n+\tfrac{1}{2})t)(He_{n+1}(x)\mathrm{e}^{-x^2/4} \\ &\qquad\qquad + He_n(x)Q_1^{-1}(x)\mathrm{e}^{-3x^2/4}) \\ &n = 0, 1, 2, \ldots. \end{aligned}$$

The functions $\theta_i(x,t)$ are orthonormal stationary solutions of the Schrödinger equation for the new potential. Note that, in contrast to [1, 2], we give the normalization constant for the ground-state function.

Consider first the action of the integral transformation operator (23) on the oscillator coherent states (30). After some simple manipulations we obtain

$$\varphi_z(x,t) = M\psi_z(x,t) = (2\pi)^{-1/4}(z\zeta)^{-1/2}\exp(-\tfrac{1}{2}zz^* - \tfrac{1}{4}x^2) \times \left[\exp(zx - \tfrac{1}{2}z^2) + \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{z-x}{\sqrt{2}}\right)Q_1^{-1}(x)\right] \qquad \zeta \neq 0.$$

The direct calculation of the norm of this function gives the following result:

$$\langle\varphi_z|\varphi_z\rangle = (zz^*)^{-1}\left(\frac{\exp(-zz^*)}{C^2-1} + 1\right).$$

One can expand this function on the set $\{\Theta_i(x,t)\}$. The expansion (31) is very useful for this purpose. The expansion coefficient at $\Theta_0(x,t)$ must be calculated directly, while the others are found by acting on expansion (31) by the transformation operator M. Taking into account the value of the integral

$$\langle\Theta_0|\varphi_z\rangle = (2\pi)^{-1/4}C\zeta^{-1}(C^2-1)^{-1/2}\exp(-\tfrac{1}{2}zz^*)$$

we obtain the following expansion:

$$\varphi_z(x,t) = \exp(-\tfrac{1}{2}zz^*)\left[\sqrt{z\zeta}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(n+1)!}}\Theta_{n+1}(x,t) + \frac{C}{\sqrt{C^2-1}}\Theta_0(x,t)\right].$$

Finally, taking into account the norm of this function, we obtain the coherent states for isospectral oscillator Hamiltonians, normalized to unity:

$$\Theta_z(x,t) = C[1 + (C^2-1)\exp(zz^*)]^{-1/2}\left[\Theta_0(x,t) + \frac{\sqrt{C^2-1}}{C}\sqrt{\frac{\zeta}{z}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n!}}\Theta_n(x,t)\right]. \tag{33}$$

Note that after being normalized, this function accepts for ζ the value $\zeta = 0$ and, at $\zeta = 0$, it coincides with the ground-state function of the new Hamiltonian. Another limiting case, $C \to \infty$, corresponds to the harmonic oscillator potential with the energy spectrum shifted to unity. It can readily be seen that in this case the functions $\Theta_z(x,t)$, in contrast to the coherent states obtained in [2], turn into the harmonic oscillator coherent states.

Expansion (33) permits one to calculate the energy expectation value for the coherent state $\Theta_z(x,t)$:

$$\langle E\rangle_z = \frac{zz^*(C^2-1)}{\exp(-zz^*) + C^2 - 1} - \frac{1}{2}.$$

The differential transformation operator (10) gives another system of coherent states:

$$\overline{\varphi}_z(x,t) = (2\pi)^{-1/4}\exp(-\tfrac{1}{2}zz^* - \tfrac{1}{2}z^2 + zx - \tfrac{1}{4}x^2 - \tfrac{\mathrm{i}}{2}t) \times \left(z - x - \sqrt{\frac{2}{\pi}}\frac{\exp(-x^2/2)}{C + \mathrm{erf}\left(x/\sqrt{2}\right)}\right). \tag{34}$$

For $\zeta = 0$ this function reduces to the first excited-state function of the new Hamiltonian, which has the property $\overline{b}\Theta_1 = 0$.

It is not difficult to analyse the completeness of the set of coherent states by the same means as in [1, 2].

3.2.2. After this work was mailed to this journal, [2] appeared in the literature and this section appears as a result of the referee's comments.

It is very easy, with the help of formula (31), to obtain the expansion of function (34) on the set of $\{\Theta_i(x,t)\}$. This expansion completely coincides with formula (5.13) of [2]. We can conclude that the differential transformation operator gives the same coherent states as that of the 'w-distortion' technique of [2] at the particular value of distortion parameter $w = 2$ when the displacement operator is used to obtain the coherent states at $t = 0$. The time evolution of these states is described by the usual propagator operator technique. Comparison also shows that the series (5.13) of [2] in a coordinate representation converges to the sufficiently simple function $\overline{\Theta}_z(x,0) = \exp(-zz^*/2)(1+zz^*)^{-1/2}\overline{\varphi}_z(x,0)$. Parameter λ of [2] is connected with our C by the relation $\lambda = \sqrt{\pi}/2C$.

Since the 'w-distorted' coherent states (4.4) of [2] do not contain the ground-state function Θ_0 in their decomposition on the proper functions of the new Hamiltonian, they are orthogonal to Θ_0. We can conclude from this that it is impossible to reproduce by means of the 'distortion' technique the coherent states Θ_z obtained by the integral transformation operator M.

3.2.3. Consider now the following solutions of the initial Schrödinger equation as transformation functions:

$$u_k(x,t) = P_{2k}(x)\exp(x^2/4 + \mathrm{i}(2k+\tfrac{1}{2})t) \qquad P_k(x) = (-\mathrm{i})^k He_k(\mathrm{i}x)$$
$$k = 0, 1, 2, \ldots.$$

Note that $H_0 u_k(x,t) = -(2k+\frac{1}{2})u_k(x,t)$. These transformation functions produce the family of exactly solvable stationary potentials previously obtained in [11]:

$$V_1(x,t) \equiv V^{(2k)}(x) = -\frac{x^2}{4} + 4k(2k-1)\frac{P_{2k-2}(x)}{P_{2k}(x)} - 8k^2\left(\frac{P_{2k-1}(x)}{P_{2k}(x)}\right)^2 + 1. \tag{35}$$

The discrete spectrum of the new Hamiltonians with potentials (35) consists of an equidistant part, $E_{n+1} = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$, and a separately disposed ground-state level $E_0^{(2k)} = -2k - \frac{1}{2}$. The wavefunctions of the stationary states of these potentials can be obtained both with the help of differential transformation (10) and with the help of integral transformation (23), whose action on the functions $\psi_n(x,t)$ is distinguished by a constant factor unessential in this case:

$$\varphi_{n+1}(x,t) = L\psi_n(x,t) = (2\pi)^{-1/4}(n!)^{-1/2}\exp(-x^2/4 - \mathrm{i}(n+\tfrac{1}{2})t)$$
$$\times[He_n(x)P_{2k+1}(x)/P_{2k}(x) - nHe_{n-1}(x)]. \tag{36}$$

The ground-state function of the new Hamiltonian is found using formula (11):

$$\varphi_0(x,t) = \exp(-x^2/4 + \mathrm{i}(2k+\tfrac{1}{2})t)P_{2k}^{-1}(x). \tag{37}$$

The symmetry operator L^+L is a linear function of the harmonic oscillator Hamiltonian, $L^+L = H_0 + 2k + \frac{1}{2}$ and LL^+ is the same function of the anharmonic oscillator Hamiltonian with potential (35), $LL^+ = H^{(2k)} + 2k + \frac{1}{2}$. This property permits one to easily calculate the normalization constants of functions (36):

$$\langle\varphi_{n+1}|\varphi_{n+1}\rangle = \langle\psi_n|L^+L|\psi_n\rangle = 2k + n + 1.$$

The normalization constant for the ground-state function (37) is calculated by direct integration:

$$\langle\varphi_0|\varphi_0\rangle = \sqrt{2\pi}/(2k)!.$$

For $k = 1$, these functions are investigated in detail in [23]. However, we believe that formula (4.19) from this work is incorrect, and, as a result, the given normalization constants are wrong. For $k = 2$, functions (36) and (37) are investigated in detail in [24].

We shall designate the stationary-state functions of Hamiltonian $H^{(2k)}$, normalized to unity as Θ_n, $n = 0, 1, 2, \ldots$. They form an orthonormal basis set in the space $L_1^2(R)$.

To obtain the coherent states of the new Hamiltonians, we must act on the harmonic oscillator coherent states by the operator L or the operator M. The operator M, being applied to these functions gives, to a constant factor, the following result:

$$\tilde{\varphi}_z(x,t) \equiv \varphi_z(x,t) = \exp(-x^2/4 - zz^*/2 - \mathrm{i}(2k+\tfrac{1}{2})t)\left(\frac{P'_{2k}(x)}{P_{2k}(x)} + x - \frac{\partial}{\partial x}\right)\frac{\partial^{2k}}{\partial x^{2k}} I(z,x) \tag{38}$$

where

$$I(z,x) = \sqrt{\frac{\pi}{2}}\exp(x^2/2)\,\mathrm{erf}\left(\frac{z-x}{\sqrt{2}}\right).$$

Decomposition (31) permits one to obtain the decomposition of function (38) on the set of states $\{\Theta_i\}$:

$$\varphi_z(x,t) = (2\pi)^{1/4}\mathrm{e}^{-zz^*/2}\left[\mathrm{e}^{-\mathrm{i}t/2}\zeta^{2k+1}\sum_{s=0}^{\infty}\frac{z^s}{\sqrt{(2k+s+1)s!}}\Theta_{s+1} + \sqrt{(2k)!}\Theta_0\right].$$

Note, that for $\zeta = 0$ function (31) coincides with the ground-state function Θ_0. Using decomposition (31), we calculate the normalizing integral of the function φ_z,

$$\langle\varphi_z|\varphi_z\rangle = (2\pi)^{1/2}(2k)!\sum_{s=0}^{2k}\frac{(-1)^s}{s!}(zz^*)^s$$

and the energy expectation value,

$$\langle\varphi_z|H^{(2k)}|\varphi_z\rangle = (2\pi)^{1/2}\left[(zz^*)^{2k+1} - (2k)!(2k+\tfrac{1}{2})\sum_{s=0}^{2k}\frac{(-1)^s}{s!}(zz^*)^s\right]. \tag{39}$$

Function (38) is expressed through the probability integral $\mathrm{erf}(z)$. However, their explicit expression contains only elementary functions. In particular, for $k = 0$ we have the harmonic oscillator with the origin of the energy axis shifted upwards for unity. All formulae obtained in this case go into the corresponding harmonic oscillator formulae, for example, from (39) we obtain $\langle\varphi_z|H^{(0)}|\varphi_z\rangle = (2\pi)^{1/2}(zz^* - \frac{1}{2})$.

For $k = 1$ function (38) has the form

$$\varphi_z(x,t) = N_z\left(z^2 + \frac{2(1-zx)}{1+x^2}\right)\exp(-\tfrac{1}{4}x^2 + zx - \tfrac{1}{2}zz^* - \tfrac{1}{2}z^2 + \tfrac{5}{2}\mathrm{i}t) \tag{40}$$

with

$$N_z = (2\pi)^{-1/4}(1 + (1-zz^*)^2)^{-1/2}.$$

The coordinate and momentum expectation values in states (40) are expressed via the probability integral, in particular

$$\langle x\rangle = z + z^* + \sqrt{2\pi}(1 + (1-zz^*)^2)^{-1}\exp(\tfrac{1}{2} - \tfrac{1}{2}(z+z^*)^2) \times \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}(z+z^*)}(\mathrm{i} - z - z^* - \mathrm{i}zz^*)\,\mathrm{erfc}\,\frac{1 - \mathrm{i}z - \mathrm{i}z^*}{\sqrt{2}}\right].$$

The differential transformation (10) leads to another system of coherent states; for instance, for $k = 1$ we obtain

$$\overline{\varphi}_z(x,t) = L\psi_z(x,t) = (2\pi)^{-1/4}\left(z - x - \frac{2x}{1+x^2}\right)\exp(-\tfrac{1}{4}x^2 + zx - \tfrac{1}{2}zz^* - \tfrac{1}{2}z^2 - \tfrac{1}{2}\mathrm{i}t).$$

These functions at $\zeta = 0$ coincide with the first excited-state function of Hamiltonian $H^{(2)}$ which has the property $\overline{b}\Theta_1 = 0$.

4. Conclusion

The transformations introduced here can also be used for the construction of time-dependent exactly solvable potentials.

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